**1 a i** $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & d^2+bc \end{bmatrix}$

**ii** $3A = 3\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$

**b i**

$$A^2 = 3A - I$$

$$\begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & d^2+bc \end{bmatrix} = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & d^2+bc \end{bmatrix} = \begin{bmatrix} 3a-1 & 3b \\ 3c & 3d-1 \end{bmatrix}$$

Equating the top right entries gives $ab+bd = 3b$. Since $b \neq 0$, this implies that

$$a + d = 1.$$

**ii** If $A^2 = 3A - I$ then

$$3A - A^2 = I$$
$$A(3I - A) = I$$

so this implies that $A$ has an inverse and that $A^{-1} = 3I - A$. Therefore,

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & -d \end{bmatrix} = \begin{bmatrix} 3-a & -b \\ -c & 3-d \end{bmatrix}$$

Equating top right entries gives

$$\frac{-b}{ad-bc} = -b.$$

Since $b \neq 0$, this implies that

$$\det(A) = ad - bc = 1.$$

**c** Since $ad - bc = 1$, we know that $A^{-1}$ exists and that

$$A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Now since $a + d = 3$, we know that

$$3I - A = \begin{bmatrix} 3-a & -b \\ -c & 3-d \end{bmatrix}$$
$$= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad = A^{-1}.$$

Therefore,

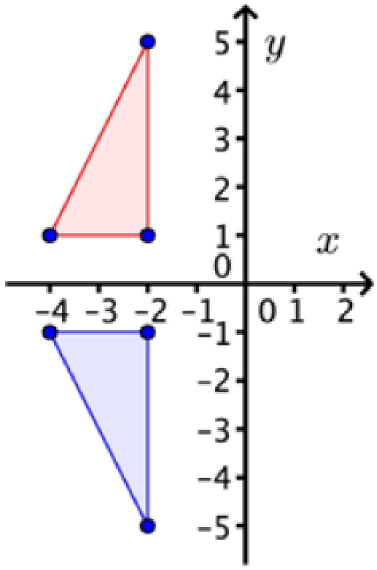
$$A(3I - A) = I$$
$$3A - A^2 = I$$
$$A^2 = 3A - I,$$

as required.

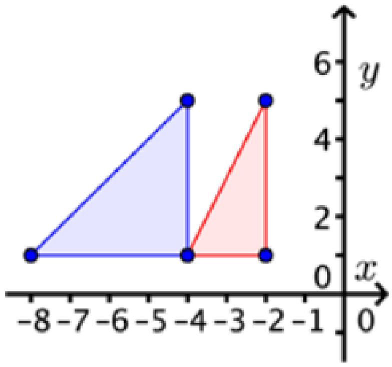
**2 a** This is a translation $6$ units to the right and $3$ units up, so the transformation is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} x+6 \\ y+3 \end{bmatrix}$$

**b**




**c**




**d** The required transformation is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} x-3 \\ 2y+2 \end{bmatrix}$$

Therefore, $x' = x - 3$ and $y' = 2y + 2$. Solving for $x$ and $y$ gives,

$$x = x' + 3 \text{ and } y = \frac{y'-2}{2}.$$

Substituting these into the equation $y = x^2$ gives,

$$\begin{aligned} \frac{y'-2}{2} &= (x'+3)^2 \\ y' - 2 &= 2(x'+3)^2 \\ y' &= 2(x'+3)^2 + 2 \end{aligned}$$

so that the image has equation $y = 2(x+3)^2 + 2$.

**e** This function can be obtained by a sequence of $3$ transformations:

**f** a dilation by a factor of $2$ from the $x$-axis then,

**g** a reflection in the $x$-axis then,

**h** a translation by the vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

The required transformation has rule,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} x+3 \\ -2y+4 \end{bmatrix}.$$

**3 a** The transformation has rule $(x, y) \to (4x, y)$ so that

$$(1,1) \to (4,1).$$

**b i** The image of the square is now a rectangle with vertices $A'(0,0), B'(0,1), C'(4,1), E'(4,0)$.

**ii** The area of square $ABCE$ is clearly $1$.

**iii** The area of its image $A'B'C'E'$ is $4$.

**iv** The area of the image would be $k$.

**c** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4x \\ y \end{bmatrix}.$

**d i** Since $x' = 4x$ and $y' = y$ we know that

$$x = \frac{x'}{4} \text{ and } y = y'.$$

Substituting these values into $y = x^2$ gives,

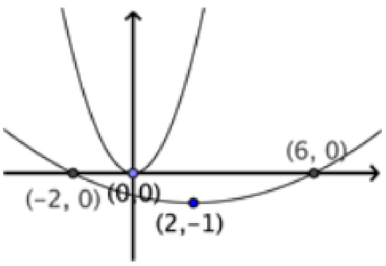
$$y' = \left(\frac{x'}{4}\right)^2$$

Therefore the equation of the image is $y = \dfrac{1}{16}x^2$.

**ii** A translation by vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ corresponds to a translation by $2$ units in the$x$-direction and $-1$ units in the $y$-direction. The equation will therefore nowbe

$$y = \frac{1}{16}(x-2)^2 - 1.$$

**iii**




**e** This function can be obtained by a sequence of $2$ transformations:

**f** a translation by the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then

**g** a dilation by a factor of $\frac{1}{5}$ from the $x$-axis. Therefore, the required transformation is,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{bmatrix} \left( \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right)$$

$$= \begin{bmatrix} x+2 \\ \frac{1}{5}(y+3) \end{bmatrix}.$$

**4 a** The matrix corresponds to a rotation by angle,

$$\theta = \cos^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{4}{5}\right).$$

**b** **i** The circle has centre $(0, 1)$ and radius $1$. Therefore its equation is

$$x^2 + (y-1)^2 = 1 \qquad (1)$$

**ii** The rotation will change the centre of the circle, but not its radius. Tofind the image of the centre, we evaluate,

$$\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}.$$

Therefore, the circle has centre $\left(-\frac{4}{5}, \frac{3}{5}\right)$ and equation,

$$\left(x + \frac{4}{5}\right)^2 + \left(y - \frac{3}{5}\right)^2 = 1 \qquad (2)$$

**c** Expanding and simplifying equations $(1)$ and $(2)$ gives,

$$x^2 + y^2 - 2y = 0 \qquad (3)$$
$$x^2 + \frac{8x}{5} + y^2 + \frac{6y}{5} = 0 \qquad (4)$$

Subtract $(4)$ from $(3)$ to give $y = 2x$. Substitute $y = 2x$ equation $(3)$ to obtain

$$\begin{aligned} x^2 + (4x^2) - 4x &= 0 \\ 5x^2 - 4x &= 0 \\ x(5x-4) &= 0 \\ x &= 0, \frac{4}{5} \\ y &= 0, \frac{8}{5}. \end{aligned}$$

so that $(0, 0)$ and $\left(\frac{4}{5}, \frac{8}{5}\right)$ are the required points.

**5** **a** $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -3 \\ 11 \end{bmatrix}$ Therefore, $(-2, 5) \to (-3, 11)$.

**b**

$$\begin{aligned} M^{-1} &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{4(3) - (1)(2)} \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix} \\ &= \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}. \end{aligned}$$

**c** If $(11, 13)$ is the image of point $(a, b)$ then,

$$\begin{aligned} \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 11 \\ 13 \end{bmatrix} \\ \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 11 \\ 13 \end{bmatrix} \\ &= \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 11 \\ 13 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 3 \end{bmatrix} \end{aligned}$$

**d** $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} 5a \\ 5a \end{bmatrix}$

Therefore, $(a, a) \to (5a, 5a)$

**e** $$M \begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} 4a+b \\ 2a+3b \end{bmatrix} = \begin{bmatrix} \lambda a \\ \lambda b \end{bmatrix}$$

We obtain simultaneous equations

$$(4-\lambda)a + b = 0 \quad (1)$$

$$2a + (3-\lambda)b = 0. \quad (2)$$

Eliminating $a$ gives,

$$(\lambda-2)(\lambda-5)b = 0.$$

Since $b \neq 0$, this implies that $\lambda = 2$ or $\lambda = 5$.

**Case 1.** If $\lambda = 2$ then equations $(1)$ and $(2)$ are equivalent. Therefore,

$$\begin{aligned} 2a + b &= 0 \\ b &= -2a. \end{aligned}$$

**Case 1.** If $\lambda = 5$ then equations $(1)$ and $(2)$ are also equivalent. Therefore,

$$\begin{aligned} -a + b &= 0 \\ b &= a. \end{aligned}$$

**6**

**a** $R = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

**b** The inverse matrix will simply be a rotation matrix by $\frac{\pi}{4}$ in the clockwise direction,

$$R^{-1} = \begin{bmatrix} \cos(-45^\circ) & -\sin(-45^\circ) \\ \sin(-45^\circ) & \cos(-45^\circ) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

**c** The point $A(a,b)$ is rotated by $\frac{\pi}{4}$ anticlockwise to the point $A'(1,1)$. Since $OA'$ is at an angle $\frac{\pi}{4}$ to the $x$-axis, the original point $A$ must be on the $x$-axis. Moreover, since

$$OA' = \sqrt{1^2+1^2} = \sqrt{2},$$

we know that $OA = \sqrt{2}$. Therefore the required coordinates are $A(\sqrt{2}, 0)$.

**d**
$$\begin{aligned} R\begin{bmatrix} c \\ d \end{bmatrix} &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \begin{bmatrix} c \\ d \end{bmatrix} &= R^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}. \end{aligned}$$

**e i**
$$\begin{aligned} R\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} x' \\ y' \end{bmatrix} \\ \begin{bmatrix} x \\ y \end{bmatrix} &= R^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' \\ -\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' \end{bmatrix}$$

**ii** Since

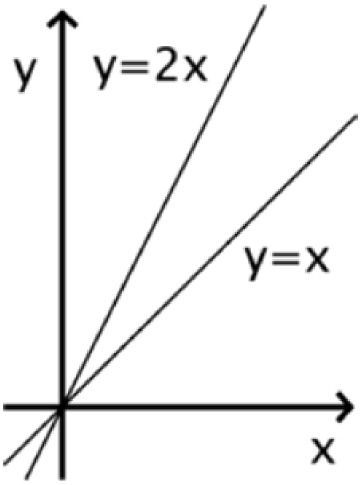
$$x = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$$
$$y = -\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$$

The image of $y = x^2$ is

$$-\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' = (\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y')^2$$
$$\sqrt{2}(y' - x') = (x' + y')^2.$$

Therefore the image has equation $\sqrt{2}(y - x) = (x + y)^2$.

**7 a**




**b** The acute angle between the $x$-axis and $y = x$ is $\theta_1 = \frac{\pi}{4}$. The acute angle between the $x$-axis and the line $y = 2x$ is $\theta_2 = \tan^{-1} 2$. The acute angle between the two lines will then be

$$\theta = \tan^{-1} 2 - \frac{\pi}{4}.$$

That is $a = 2$ and $b = \frac{\pi}{4}$.

**c** We need to evaluate $\cos\theta$ and $\sin\theta$. We have

$$\begin{aligned} \cos\theta &= \cos(\tan^{-1} 2 - \frac{\pi}{4}) \\ &= \cos(\tan^{-1} 2)\cos\frac{\pi}{4} + \sin(\tan^{-1} 2)\sin\frac{\pi}{4} \\ &= \frac{1}{\sqrt{5}}\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{5}}\frac{1}{\sqrt{2}} \\ &= \frac{3}{\sqrt{10}} \end{aligned}$$

Likewise,

$$\sin\theta = \frac{1}{\sqrt{10}}.$$

Therefore, the required matrix is

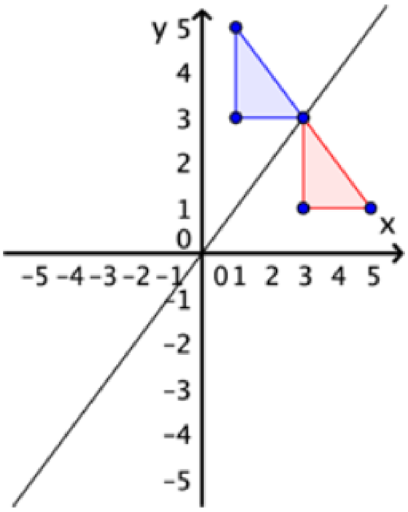
$$\begin{bmatrix} \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}.$$

**8** **a** **i** $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Therefore, $(1,3) \to (3,1)$.

**ii** Reflecting point $(a,b)$ in the line $y = x$ simply switches the $x-$ and $y$-coordinates.Therefore the images has coordinates $A'(3,1), B'(5,1), C'(3,3)$.

**iii**




**b** **i** Since $x' = y$ and $y' = x$, the equation $y = x^2 - 2$ simply becomes $x' = (y')^2 - 2$. Ignoring the dashes, givesthe equation $x = y^2 - 2$.

**ii** Substitute $y = x$ into $y = x^2 - 2$ to give

$$\begin{aligned} x^2 - 2 &= x \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ x &= -1, 2. \end{aligned}$$

As $y = x$, the coordinates are $(-1,-1)$ and $(2,2)$

**iii** Substituting $y = x^2 - 2$ into $x = y^2 - 2$ gives,

$$\begin{aligned} (x^2-2)^2 - 2 &= x \\ x^4 - 4x^2 + 4 - 2 &= x \\ x^4 - 4x^2 - x + 2 &= 0 \end{aligned}$$

**iv** When $x = \frac{1}{2}(-1+\sqrt{5})$,

$$y = x^2 - 2 = \frac{1}{2}(-1-\sqrt{5}),$$

and when $x = \frac{1}{2}(-1-\sqrt{5})$,

$$y = x^2 - 2 = \frac{1}{2}(-1+\sqrt{5})$$

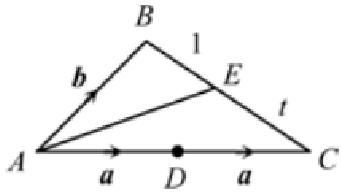
Therefore, the points of intersection are:

$$(-1,-1), (2,2), (\frac{1}{2}(-1+\sqrt{5}), \frac{1}{2}(-1-\sqrt{5})), (\frac{1}{2}(-1-\sqrt{5}), \frac{1}{2}(-1+\sqrt{5})).$$

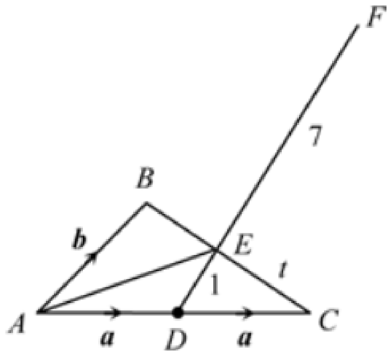
**9** **a**

$$\begin{aligned} \overrightarrow{AE} &= \overrightarrow{AC} + \overrightarrow{CE} \\ &= 2\overrightarrow{AD} + \frac{t}{t+1} \\ &= 2\boldsymbol{a} + \frac{t}{t+1}(\boldsymbol{b} - 2\boldsymbol{a}) \\ &= \frac{2(t+1)}{t+1}\boldsymbol{a} + \frac{t}{t+1}\boldsymbol{b} - \frac{2t}{t+1}\boldsymbol{a} \end{aligned}$$

$$= \frac{1}{t+1}((2t+2-2t)\boldsymbol{a} + t\boldsymbol{b})$$
$$= \frac{1}{t+1}(2\boldsymbol{a} + t\boldsymbol{b})$$




**b** $$\overrightarrow{AE} = \overrightarrow{AD} + \overrightarrow{DE}$$
$$= \boldsymbol{a} + \frac{1}{8}\overrightarrow{DF}$$
$$= \boldsymbol{a} + \frac{1}{8}(\overrightarrow{AF} - \overrightarrow{AD})$$
$$= \boldsymbol{a} + \frac{1}{8}\overrightarrow{AF} - \frac{1}{8}\boldsymbol{a}$$
$$= \frac{1}{8}(7\boldsymbol{a} + \overrightarrow{AF})$$




**c** $$\overrightarrow{AE} = \frac{1}{8}(7\boldsymbol{a} + \overrightarrow{AF})$$
$$\therefore\ 8\overrightarrow{AE} = 7\boldsymbol{a} + \overrightarrow{AF}$$
$$\therefore\ \overrightarrow{AF} = 8\overrightarrow{AE} - 7\boldsymbol{a}$$
$$= \frac{8}{t+1}(2\boldsymbol{a} + t\boldsymbol{b}) - 7\boldsymbol{a}$$
$$= \frac{1}{t+1}(16\boldsymbol{a} + 8t\boldsymbol{b} - 7(t+1)\boldsymbol{a})$$
$$= \frac{1}{t+1}(16\boldsymbol{a} + 8t\boldsymbol{b} - (7t+7)\boldsymbol{a})$$
$$= \frac{1}{t+1}((9-7t)\boldsymbol{a} + 8t\boldsymbol{b})$$
$$= \frac{9-7t}{1+t}\boldsymbol{a} + \frac{8t}{1+t}\boldsymbol{b}, \text{ as required.}$$

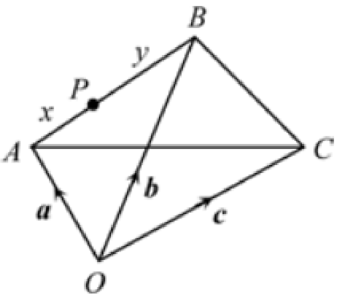
**d** If $A, B$ and $F$ are collinear, then $\overrightarrow{AF} = k\,\overrightarrow{AB}, k > 0$
$$= k\boldsymbol{b}$$
$$= 0\boldsymbol{a} + k\boldsymbol{b}$$

$$\therefore\ \frac{9-7t}{1+t} = 0$$
$$\therefore\ 9 - 7t = 0$$
$$\therefore\ t = \frac{9}{7}$$

**10a** Assume $P$ divides $AB$ in the ratio $x : y$.
$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

$$= \boldsymbol{a} + \frac{x}{x+y}\overrightarrow{AB}$$
$$= \boldsymbol{a} + \frac{x}{x+y}(\overrightarrow{OB} - \overrightarrow{OA})$$
$$= \frac{x+y}{x+y}\boldsymbol{a} + \frac{x}{x+y}(\boldsymbol{b} - \boldsymbol{a})$$
$$= \frac{1}{x+y}((x+y-x)\boldsymbol{a} + x\boldsymbol{b})$$
$$= \frac{y}{x+y}\boldsymbol{a} + \frac{x}{x+y}\boldsymbol{b}$$
$$= m\boldsymbol{a} + n\boldsymbol{b} \text{ where } m = \frac{y}{x+y},\ n = \frac{x}{x+y}, m, n \geq 0$$

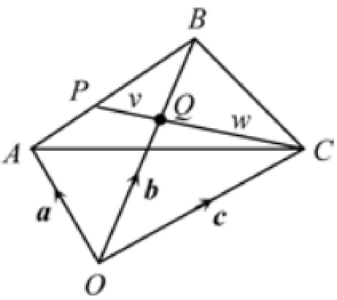
and $m + n = \frac{y}{x+y} + \frac{x}{x+y}$
$= 1$, as required.




**b** $\overrightarrow{PC} = -\overrightarrow{AP} - \overrightarrow{OA} + \overrightarrow{OC}$
$$= -n(\boldsymbol{b} - \boldsymbol{a}) - \boldsymbol{a} + \boldsymbol{c}$$
$$= -n\boldsymbol{b} + n\boldsymbol{a} - \boldsymbol{a} + \boldsymbol{c}$$
$$= (n-1)\boldsymbol{a} - n\boldsymbol{b} + \boldsymbol{c}$$

**c** Assume $Q$ divides $PC$ in the ratio $v : w$.

$\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AP} + \overrightarrow{PQ}$
$$= \boldsymbol{a} + n(\boldsymbol{b} - \boldsymbol{a}) + \frac{v}{v+w}\overrightarrow{PC}$$
$$= \boldsymbol{a} + n\boldsymbol{b} - n\boldsymbol{a} + \frac{v}{v+w}((n-1)\boldsymbol{a} - n\boldsymbol{b} + \boldsymbol{c})$$
$$= \frac{1}{v+w}((v+w)\boldsymbol{a} + n(v+w)\boldsymbol{b} - n(v+w)\boldsymbol{a} + v(n-1)\boldsymbol{a} - nv\boldsymbol{b} + v\boldsymbol{c})$$
$$= \frac{1}{v+w}((v+w-nv-nw+vn-v)\boldsymbol{a} + (nv+nw-nv)\boldsymbol{b} + v\boldsymbol{c})$$
$$= \frac{1}{v+w}((w-nw)\boldsymbol{a} + nw\boldsymbol{b} + v\boldsymbol{c})$$
$$= \frac{w(1-n)}{v+w}\boldsymbol{a} + \frac{nw}{v+w}\boldsymbol{b} + \frac{v}{v+w}\boldsymbol{c}$$
$$= \lambda\boldsymbol{a} + \mu\boldsymbol{b} + \gamma\boldsymbol{c}$$




where $\lambda = \frac{w(1-n)}{v+w}, \mu = \frac{nw}{v+w}, \gamma = \frac{v}{v+w}, \lambda, \mu, \gamma \geq 0$

and $\lambda + \mu + \gamma = \dfrac{w(1-n) + nw + v}{v+w}$

$$= \frac{w - nw + nw + v}{v+w}$$

$$= \frac{v+w}{v+w} = 1, \text{as required.}$$

**11a** $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= \boldsymbol{b} - \boldsymbol{a}$$

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OB} + \overrightarrow{BQ}$$

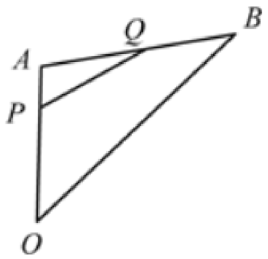
$$= -\overrightarrow{OP} + \boldsymbol{b} - \overrightarrow{QB}$$

$$= -\frac{4}{5}\overrightarrow{OA} + \boldsymbol{b} - \frac{1}{2}\overrightarrow{AB}$$

$$= -\frac{4}{5}\boldsymbol{a} + \boldsymbol{b} - \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$$

$$= \left(\frac{-4}{5} + \frac{1}{2}\right)\boldsymbol{a} + \left(1 - \frac{1}{2}\right)\boldsymbol{b}$$

$$= \frac{-3}{10}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$$




**b**

**i** $\overrightarrow{QR} = n\,\overrightarrow{PQ}$

$$= n\left(\frac{-3}{10}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}\right)$$

**ii** $\overrightarrow{QR} = \overrightarrow{PR} - \overrightarrow{PQ}$

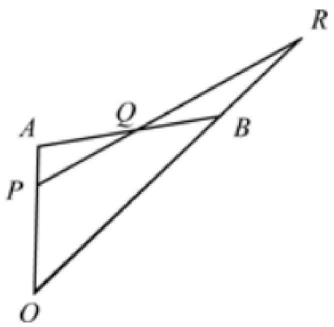
$$= (\overrightarrow{OR} - \overrightarrow{OP}) - \overrightarrow{PQ}$$

$$= (\overrightarrow{OB} + \overrightarrow{BR}) - \overrightarrow{OP} - \overrightarrow{PQ}$$

$$= \boldsymbol{b} + k\boldsymbol{b} - \frac{4}{5}\boldsymbol{a} - \left(\frac{-3}{10}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}\right)$$

$$= \left(\frac{3}{10} - \frac{4}{5}\right)\boldsymbol{a} + \left(k + 1 - \frac{1}{2}\right)\boldsymbol{b}$$

$$= -\frac{1}{2}\boldsymbol{a} + \left(k + \frac{1}{2}\right)\boldsymbol{b}$$

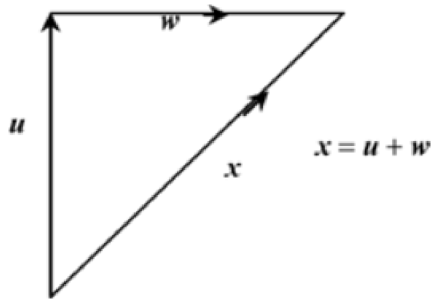


**c** $n\left(\frac{-3}{10}\boldsymbol{a}+\frac{1}{2}\boldsymbol{b}\right)=-\frac{1}{2}\boldsymbol{a}+\left(k+\frac{1}{2}\right)\boldsymbol{b}$

$$\therefore \frac{-3}{10}n=-\frac{1}{2}$$
$$\therefore n=-\frac{1}{2}\times\frac{-10}{3}=\frac{5}{3}$$
$$\text{and } \frac{1}{2}n=k+\frac{1}{2}$$
$$\therefore k=\frac{1}{2}\times\frac{5}{3}-\frac{1}{2}=\frac{1}{3}$$

**12a** Let $\boldsymbol{x}$ be the (proper) velocity of the wind relative to a stationary object.
Let $\boldsymbol{u}$ be the man's velocity, $4$ km in a northerly direction.

Let $\boldsymbol{w}$ be the apparent velocity of the wind.




When the man doubles his speed the wind appears to come from the north west.

Let $\boldsymbol{w}'$ be the new apparent velocity of the wind.

The new velocity is $2\boldsymbol{u}=\boldsymbol{u}+\boldsymbol{u}$.

The second vector diagram is superimposed on the first.

The vertices are labelled to describe the triangles.

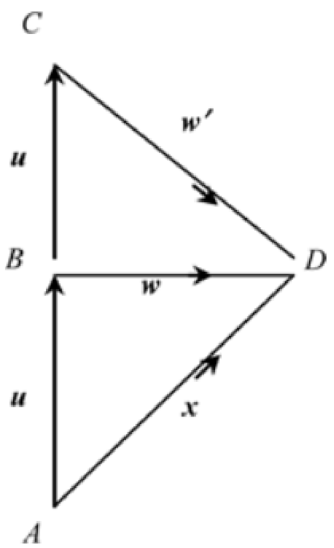
The triangle $BCD$ is isosceles as $\angle CBD$ is a right angle and $\angle BCD=45^\circ$.

$|\boldsymbol{u}|=4$ and therefore $|\boldsymbol{w}|=4$.

By Pythagoras' theorem,

$$|\boldsymbol{x}|^2=4^2+4^2$$
$$\text{and so, } |\boldsymbol{x}|=4\sqrt{2}$$

and the direction that it blows from is south west.

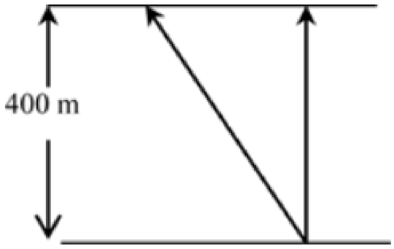



**b** The northerly component of the swimmer's velocity is $2000$ m/h.
The river is $400$ m wide.

It takes $\frac{400}{2000} = \frac{1}{5}$ hour to reach the north bank.

The river is flowing from east to west at $1 \text{ km/h} = 1000 \text{ m/h}$.

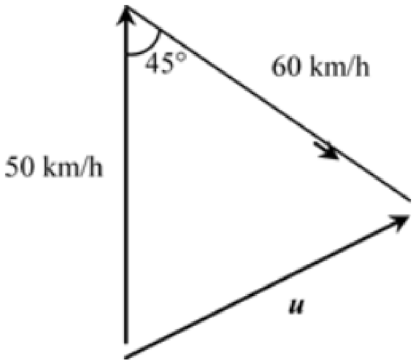
Hence in 1hour the swimmer has gone $\frac{1}{5} \times 1000 = 200 \text{ m}$ downstream.




**c** Let $\boldsymbol{u}$ be the true velocity of the wind.
The cosine rule can be used to determine the magnitude of the velocity.

$$\begin{aligned}|\boldsymbol{u}|^2 &= 50^2 + 60^2 - 2 \times 50 \times 60 \cos 45^\circ \\ &= 2500 + 3600 - 6000 \cos 45^\circ \\ &= 6100 - 3000\sqrt{2}\end{aligned}$$

So $|\boldsymbol{u}| = 43.1 \text{ km/h}$ (correct to one decimal place).




For the direction to be determined the sine rule is used.

$$\begin{aligned}\frac{60}{\sin\alpha} &= \frac{|\boldsymbol{u}|}{\sin 45^\circ} \\ \therefore \sin\alpha &= \frac{60 \sin 45^\circ}{|\boldsymbol{u}|}\end{aligned}$$

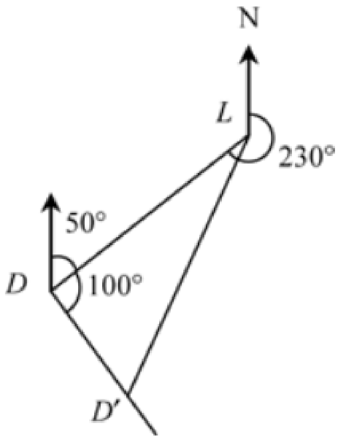
$$\text{Therefore, } \alpha = 79.88^\circ$$

The true velocity of the wind is $43.1 \text{ km/h}$ blowing at a bearing of $080^\circ$ (correct to the nearest degree).

**d** Let $\angle DLD' = \alpha^\circ$.
From the diagram and using the sine rule

$$\begin{aligned}\frac{5}{\sin\alpha} &= \frac{35}{\sin 100^\circ} \\ \therefore \sin\alpha &= \frac{5 \sin 100^\circ}{35} \\ \therefore \alpha &\approx 8.1^\circ\end{aligned}$$

This represents a bearing of $230^\circ - 8.1^\circ = 221.9^\circ$, or $222^\circ$ correct to the nearest degree.



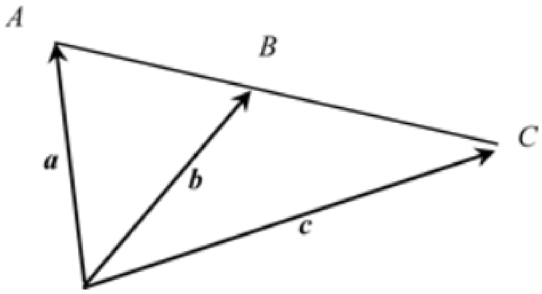

**13a** $\overrightarrow{OA} = \boldsymbol{a}, \overrightarrow{OB} = \mathbf{b}, \overrightarrow{OC} = \boldsymbol{c}$

Since $A, B$ and $C$ are collinear,

$\overrightarrow{AC} = k\overrightarrow{AB}$.

$$\begin{aligned} \boldsymbol{c} &= \overrightarrow{OC} \\ &= \overrightarrow{OA} + \overrightarrow{AC} \\ &= \overrightarrow{OA} + k\overrightarrow{AB} \\ &= \overrightarrow{OA} + k(\overrightarrow{AO} + \overrightarrow{OB}) \\ &= (1-k)\boldsymbol{a} + k\mathbf{b} \end{aligned}$$

So if $\boldsymbol{c} = \alpha\boldsymbol{a} + \beta\mathbf{b}$, and $O$ does not lie on the line $ABC$, then $\alpha = 1 - k$ and $\beta = k$ and $\alpha + \beta = 1$.




**b i** Let $N$ be the midpoint of $YZ$.

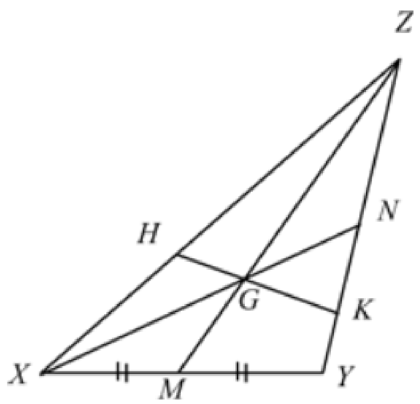
As $G$ lies on $ZM$,

$\overrightarrow{ZG} = k\overrightarrow{ZM}$ for some non-zero real number $k$.

Similarly,

$\overrightarrow{XG} = l\overrightarrow{XN}$ for some non-zero real number $l$.

$\overrightarrow{ZG}$ will be found in two different ways to obtain simultaneous equations in$k$ and $l$.



$$\begin{aligned}\overrightarrow{ZG} &= k\overrightarrow{ZM}\\ &= k(\overrightarrow{ZX}+\overrightarrow{XM})\\ &= k\overrightarrow{ZX}+\frac{1}{2}k\overrightarrow{XY}\\ &= k\overrightarrow{ZX}+\frac{1}{2}k(\overrightarrow{XZ}+\overrightarrow{ZY})\\ &= \frac{1}{2}k\overrightarrow{ZX}+\frac{1}{2}k\overrightarrow{ZY}\end{aligned}$$

$$\begin{aligned}\text{Also } \overrightarrow{ZG} &= \overrightarrow{ZX}+\overrightarrow{XG}\\ &= \overrightarrow{ZX}+l\overrightarrow{XN}\\ &= \overrightarrow{ZX}+l(\overrightarrow{XZ}+\overrightarrow{ZN})\\ &= \overrightarrow{ZX}+l\left(-\overrightarrow{ZX}+\frac{1}{2}\overrightarrow{ZY}\right)\\ &= (1-l)\overrightarrow{ZX}+\frac{1}{2}l\overrightarrow{ZY}\end{aligned}$$

$$\text{Thus } \overrightarrow{ZG} = \frac{1}{2}k\overrightarrow{ZX}+\frac{1}{2}k\overrightarrow{ZY} = (1-l)\overrightarrow{ZX}+\frac{1}{2}l\overrightarrow{ZY}$$

is not parallel, to $\overrightarrow{ZY}$

Hence equating coefficients,

$\frac{1}{2}k = 1-l$ and $\frac{1}{2}k = \frac{1}{2}l$

$\therefore\ l = k \text{ and } \frac{1}{2}k = 1-k$

$$\therefore\ k = l = \frac{2}{3}$$

Thus $\overrightarrow{ZG} = \frac{2}{3}\overrightarrow{ZM}$.

**ii**

$$\begin{aligned}\overrightarrow{ZG} &= \frac{2}{3}\overrightarrow{ZM}\\ &= \frac{2}{3}(\overrightarrow{ZX}+\overrightarrow{XM})\\ &= \frac{2}{3}\overrightarrow{ZX}+\frac{1}{3}\overrightarrow{XY}\\ &= \frac{2}{3}\overrightarrow{ZX}+\frac{1}{3}(\overrightarrow{XZ}+\overrightarrow{ZY})\\ \therefore\ \overrightarrow{ZG} &= \frac{2}{3}\overrightarrow{ZX}-\frac{1}{3}\overrightarrow{ZX}+\frac{1}{3}\overrightarrow{ZY}\\ &= \frac{1}{3}\overrightarrow{ZX}+\frac{1}{3}\overrightarrow{ZY}\end{aligned}$$

But $\overrightarrow{ZH} = h\overrightarrow{ZX}$, $\overrightarrow{ZK} = k\overrightarrow{ZY}$

$\text{So } \overrightarrow{ZX} = \frac{1}{h}\overrightarrow{ZH} \text{ and } \overrightarrow{ZY} = \frac{1}{k}\overrightarrow{ZK}$

$$\text{So } \overrightarrow{ZG} = \frac{1}{3h}\overrightarrow{ZH}+\frac{1}{3k}\overrightarrow{ZK}$$

**iii** Since $H, G$ and $K$ are collinear and $\overrightarrow{ZG} = \frac{1}{3h}\overrightarrow{ZH}+\frac{1}{3k}\overrightarrow{ZK}$, from part **a**

$$\frac{1}{3h}+\frac{1}{3k} = 1$$

$$\text{and } \frac{1}{h}+\frac{1}{k} = 3$$

Hence $h=k=\frac{2}{3}$

This means $\frac{ZH}{ZX}=\frac{ZK}{ZY}$ and so triangles $ZHK$ and $ZXY$ are similar triangles and $HK$ is parallelto $XY$.

(Also $HK$ is parallel to $XY$ implies $h=k=\frac{2}{3}$.)

**v** If $h=k$ then $h=k=\frac{2}{3}$. Triangles $ZHK$ and $ZXY$ are similar.

$$\begin{aligned}\therefore \text{ Area of} \triangle ZHK &= \frac{4}{9}(\text{Area of} \triangle ZXY)\\ &= \frac{4}{9} \text{ cm }^2\end{aligned}$$

**vi** If $k=2h$,

then $\frac{1}{h}+\frac{1}{2h}=3$

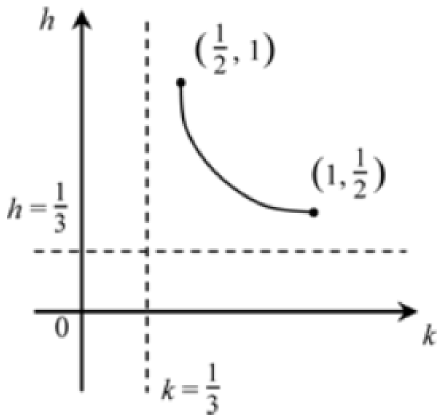
$$\therefore \frac{3}{2h}=3$$

$$\text{and } h=\frac{1}{2}$$

Thus $ZH=\frac{1}{2}ZX$ and $H$ is the midpoint of $ZX$. This means that $HG$ is the median and inthis case $K$ coincides $Y$.

**vii** If $H$ lies on the line segment $ZX$ and $K$ lies on the line segment $ZY$, then $0 \leq h \leq 1$ and $0 \leq k \leq 1$.

$$\begin{aligned}
&\text{Now} && \frac{1}{h}+\frac{1}{k}=3 && \ldots (1)\\
&\text{so} && \frac{1}{h}=3-\frac{1}{k}\\
&\therefore && \frac{1}{h}=\frac{3k-1}{k}\\
&\therefore && h=\frac{k}{3k-1} && \ldots (2)\\
&\text{From (1),} && \frac{1}{k}=3-\frac{1}{h}\\
& && =\frac{3h-1}{h}\\
&\therefore && k=\frac{h}{3h-1} && \ldots (3)\\
&\text{Now} && 0<h \leq 1\\
&\therefore \text{ from (2),} && 0<\frac{k}{3k-1} \leq 1\\
&\therefore && 0<k \leq 3k-1\\
&\text{Consider} && 3k-1 \geq k\\
&\therefore && 2k \geq 1\\
&\therefore && k \geq \frac{1}{2}\\
&\text{Hence} && \frac{1}{2} \leq k \leq 1\\
&\text{Similarly} && 0<k \leq 1\\
&\therefore \text{ from (3),} && 0<\frac{h}{3h-1} \leq 1\\
&\therefore && h \geq \frac{1}{2}\\
&\text{Hence} && \frac{1}{2} \leq h \leq 1
\end{aligned}$$




The graph of $h$ against $k$ is part of a hyperbola as shown.

**viii** Let the area of $\triangle XYZ$ be $1 \text{ cm}^2$.

Then, as $\triangle ZKX$ and $\triangle XYZ$ have bases along $ZY$ and have the same height,

$$\frac{\text{area of } \triangle ZKX}{\text{area of } \triangle ZYX} = \frac{ZK}{ZY}$$

$$\begin{aligned} \therefore \text{ area of } \triangle ZKX &= \frac{ZK}{ZY} \times \text{ area of } \triangle XYZ \\ &= \frac{kZY}{ZY} \times 1, \text{ since } ZK = kZY \\ &= k \end{aligned}$$

Also, as $\triangle ZHK$ and $\triangle ZKX$ have bases along $ZX$ and have the same height,

$$\begin{aligned} \frac{\text{area of } \triangle ZHK}{\text{area of } \triangle ZKX} &= \frac{ZH}{ZX} \\ &= \frac{hZX}{ZX}, \text{ since } ZH = hZX \\ &= h \end{aligned}$$

$$\therefore \text{ area of } \triangle ZHK = h \times \text{ area of } \triangle ZKX$$

$$\therefore A = h\,k$$

Using equation (2) in part **vii**,

$$\begin{aligned} A &= \frac{k}{3k-1} \times k \\ &= \frac{k^2}{3k-1} \end{aligned}$$

Now, using long division, or the propFrac command of a CAS calculator, $A$ can be expressed as

$$A = \frac{1}{3}k + \frac{1}{9} + \frac{1}{9(3k-1)}$$

Thus the graph of $A$ against $k$ has an asymptote with equation $A = \frac{1}{3}k + \frac{1}{9}$.

In part **vii** it was established that $k \in \left[\frac{1}{2}, 1\right]$.

Using a CAS calculator, the minimum is at $k = \frac{2}{3}$ and then $A = \frac{4}{9}$ which appears tobe $\left(\frac{2}{3}, \frac{4}{9}\right)$

To check this algebraically, first note that for $k > \frac{1}{3}, 3k - 1 > 0$, so $A = \frac{k^2}{3k-1}$ is always positive.

Also $\quad \left(k - \frac{2}{3}\right)^2 \geq 0$

$\therefore \quad k^2 - \frac{4}{3}k + \frac{4}{9} \geq 0$

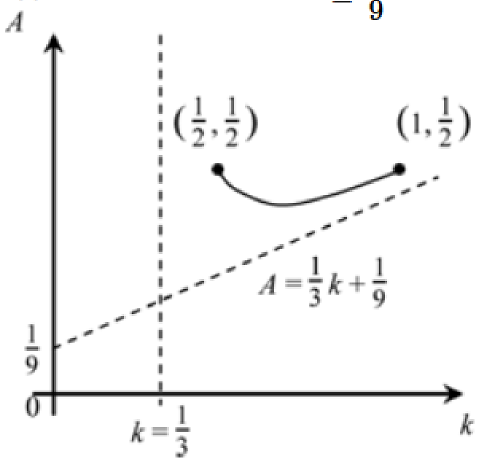
$\therefore \quad k^2 \geq \frac{4}{3}k - \frac{4}{9}$

$\therefore \quad k^2 \geq \frac{4}{9}(3k-1)$

Now $\quad A = \frac{k^2}{3k-1}$ and so

$$A \geq \frac{\frac{4}{9}(3k-1)}{3k-1}$$

$\therefore \quad A \geq \frac{4}{9}$




**14a** Let $\mathbf{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{Y} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

**i**

$$\begin{aligned} \mathrm{Tr}(\mathbf{X}+\mathbf{Y}) &= \mathrm{Tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) \\ &= \mathrm{Tr}\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \\ &= a+e+d+h \\ \mathrm{Tr}(\mathbf{X}) + \mathrm{Tr}(\mathbf{Y}) &= \mathrm{Tr}\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \mathrm{Tr}\begin{bmatrix} e & f \\ g & h \end{bmatrix} \\ &= a+d+e+h \end{aligned}$$

Hence $\mathrm{Tr}(\mathbf{X}+\mathbf{Y}) = \mathrm{Tr}(\mathbf{X}) + \mathrm{Tr}(\mathbf{Y})$

**ii**

$$\begin{aligned} \mathrm{Tr}(-\mathbf{X}) &= \mathrm{Tr}\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \\ &= -a-d \\ -\mathrm{Tr}(\mathbf{X}) &= -\mathrm{Tr}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= -(a+d) \\ &= -a-d \end{aligned}$$

Hence $\mathrm{Tr}(-\mathbf{X}) = -\mathrm{Tr}(\mathbf{X})$

**iii**

$$\begin{aligned} \mathrm{Tr}(\mathbf{XY}) &= \mathrm{Tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) \\ &= \mathrm{Tr}\begin{bmatrix} ae+bg & af+ \ bh \\ ce+dg & cf+ \ dh \end{bmatrix} \\ &= ae+bg+cf+dh \\ \mathrm{Tr}(\mathbf{YX}) &= \mathrm{Tr}\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \\ &= \mathrm{Tr}\begin{bmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{bmatrix} \end{aligned}$$

$= ae + cf + bg + dh$

Hence $\mathrm{Tr}(\mathbf{XY}) = \mathrm{Tr}(\mathbf{YX})$

**b**

$$\begin{aligned}\mathrm{Tr}(\mathbf{XY} - \mathbf{YX}) &= \mathrm{Tr}(\mathbf{XY}) - \mathrm{Tr}(\mathbf{YX}) \\ &= 0 \text{ as } \mathrm{Tr}(\mathbf{XY}) = \mathrm{Tr}(\mathbf{YX}) \\ \mathrm{Tr}(\mathbf{I}) &= \mathrm{Tr}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= 2\end{aligned}$$

As $\mathrm{Tr}(\mathbf{XY} - \mathbf{YX}) \neq \mathrm{Tr}(\mathbf{I})$

$\mathbf{XY} - \mathbf{YX} \neq \mathbf{I}$ for any $\mathbf{X}, \mathbf{Y}$